Phys 402
Fall 2022
Homework 3
Due Wednesday, 21 September @ 10 AM as a PDF upload to ELMS

1. Griffiths and Schroeter *Quantum Mechanics*, 3rd Ed., Problem 7.19 (Commutators in the spin-orbit coupled H-atom) *Hint: Make use of Griffiths Eq. [3.65]*

2. Griffiths and Schroeter *Quantum Mechanics*, 3rd Ed., Problem 7.20 (Full fine-structure formula)

3. Griffiths and Schroeter *Quantum Mechanics*, 3rd Ed., Problem 7.33 (Electron in the nucleus!)

4. Consider a perturbation problem in a 2-dimensional Hilbert space (this simplifies the math, and allows us to think about particles with spin-1/2). The unperturbed Hamiltonian is $\mathcal{H}^0 = \begin{pmatrix} 0 & \eta \\ \eta & 0 \end{pmatrix}$

a) Find the eigenvalues E_1^0, E_2^0 and eigenfunctions $|\psi_1^0\rangle$, $|\psi_2^0\rangle$ of the unperturbed Hamiltonian.

b) The perturbing Hamiltonian is $\mathcal{H}' = \begin{pmatrix} 0 & -i\lambda \\ i\lambda & 0 \end{pmatrix}$. Find the first order correction to the eigen-energies, E_1^1, E_2^1.

c) Can you supply a physical justification for the results in part (b)? Hint: Note that $\mathcal{H}^0$ is proportional to the σ_x Pauli spin matrix and $\mathcal{H}'$ is proportional to σ_y.

d) Now calculate the second order corrections to the eigen-energies, E_1^2, E_2^2. State the energies of the two eigen-states to second order in perturbation theory.

5. This problem goes through the process of quantizing the inductor-capacitor (LC) electrical resonator. Consider a parallel LC circuit, with no resistance. The inductor stores energy in magnetic fields: $U_L = \frac{1}{2}Li^2$, where i is the current through the inductor. We can change the independent variable from current i to magnetic flux Φ, as follows. The inductance is defined as the magnetic flux generated per unit current, $L = \Phi/i$, where $\Phi = \iint_S \vec{B} \cdot d\vec{A}$, and S is a surface intercepting the field created by the current. Note that the voltage drop on the inductor is given by $V(t) = L\frac{dI}{dt} = \dot{\Phi}$, where the over-dot denotes time derivative. The stored energy can be written as $U_L = \frac{1}{2}\Phi^2/L$, where we can treat the flux Φ as the new independent variable (generalized coordinate). Meanwhile the capacitor stores energy in electric field as $U_C = \frac{1}{2}CV^2 = \frac{1}{2}C\dot{\Phi}^2$. Note that the energies are now written in terms of flux Φ and its time derivative $\dot{\Phi}$.

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There is an analogy between a (mechanical) simple harmonic oscillator and the LC electrical circuit. We will actually use the dual of the traditional electrical analogy, as outlined in this table:

Mechanical	**Electrical**	**Dual Electrical**
Position x	Charge Q	Flux Φ
Mass m	Inductance L	Capacitance C
Spring constant k	$1/C$	$1/L$
Momentum $p = m\dot{x}$	$L\,i = L\dot{Q}$	$C\,\dot{\Phi}$
Kinetic Energy $\frac{1}{2}m\dot{x}^2$	$\frac{1}{2}Li^2$	$\frac{1}{2}C\dot{\Phi}^2$
Potential Energy $\frac{1}{2}kx^2$	$\frac{1}{2}Q^2/C$	$\frac{1}{2}\Phi^2/L$

a) Using the dual electrical analog of the harmonic oscillator, write the Lagrangian of the LC circuit in terms of the generalized coordinate (Φ) and its time derivative as $\mathcal{L}(\Phi, \dot{\Phi}) = T - U$, where T is the kinetic energy and U is the potential energy.

b) Find the conjugate momentum from the Lagrangian as $p = \partial\mathcal{L}/\partial\dot{\Phi}$. This is the momentum that is conjugate to the generalized coordinate Φ. What electrical quantity is conjugate to the flux?

c) Find the Hamiltonian by first re-expressing $\dot{\Phi}$ in terms of only the generalized coordinate and its conjugate momentum. Next, calculate the Hamiltonian as $\mathcal{H}(\mathrm{q}, \mathrm{p}) = \sum_{i=1}^{n} p_i\,\dot{q}_i - \mathcal{L}(\mathrm{q}_i, \dot{\mathrm{q}}_i)$, where $n = 1$ in this case, taking care to replace $\dot{\Phi}$ with it's re-expressed version.

d) Now quantize the Hamiltonian by replacing the dynamical variables with corresponding quantum operators. Because the classical dynamical variables are canonically conjugate, their corresponding quantum operators will have a non-zero commutator (recall the Poisson Bracket problem in HW 2). Write down this commutator, in addition to writing out the quantum Hamiltonian.

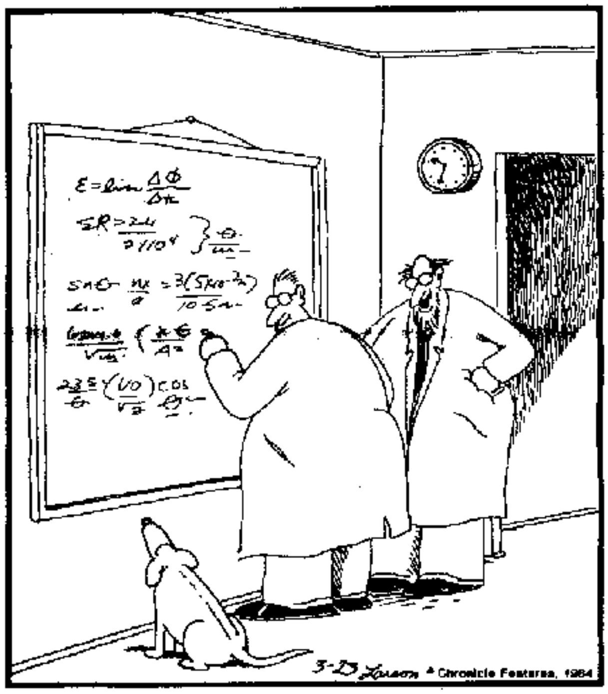
e) Express the generalized coordinate and momentum operators from part (d) in terms of raising and lowering operators, utilizing the direct analogy with the 1D quantum mechanical harmonic oscillator (take a look at section 2.3.1, and Example 2.5, of Griffiths). Write the coefficients that appear in the raising and lowering operators in terms of $\hbar$, C, and $\omega \equiv 1/\sqrt{LC}$, only.

f) Re-express the quantum LC Hamiltonian operator in terms of the raising and lowering operators, ω, and $\hbar$, only. This completes the quantization process.

EXTRA CREDIT

3. Griffiths and Schroeter *Quantum Mechanics*, 3rd Ed., Problem 7.38 (Feynman – Hellmann Theorem)

THE FAR SIDE By GARY LARSON



"Ohhhhhhh . . . Look at that, Schuster . . .
Dogs are so cute when they try to comprehend
quantum mechanics."